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On biaccessible points in the Julia set of the family $z(a+z^d)$

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Abstract

We are interested in biaccessibility in the Julia sets of polynomials with Cremer fixed points. In this paper, we consider $f_a(z) = z(a+z^d)$ where the origin is a Cremer fixed point.

D. Schleicher and S. Zakeri studied which points are biaccessible when $d = 1$ [**SZ**]. We consider when $d \geq 1$.

1 Preliminaries

In this paper, we set $f_a(z) = z(a+z^d)$ for some d greater than or equal to one. For each $0 \leq j \leq d-1$, let $\tau_j(z) = e^{2\pi i \frac{j}{d}} z$ be a $\frac{j}{d}$-rotation. Now f_a has τ_j-symmetric critical points $c_j = \tau_j(c)$, where c is one of the solutions of $a + (d+1)z^d = 0$.

Recall that the *filled Julia set* of f_a is

$$K_a = \{z \in \mathbb{C} : \{f_a^{\circ n}(z)\}_{n \geq 0} \text{ is bounded}\}$$

and the *Julia set* of f_a is $J_a = \partial K_a$. Then $f_a \circ \tau_j = \tau_j \circ f_a$ implies $\tau_j(K_a) = K_a$ and thus $\tau_j(J_a) = J_a$.

Now assume that the filled Julia set K_a is connected. Then there exists a unique conformal isomorphism:

$$\psi : \mathbb{C} - \overline{\mathbb{D}} \to \mathbb{C} - K_a$$

such that $\frac{\psi(z)}{z} \to 1$ as $z \to \infty$.

Here it is important that the following holds [Mi, Theorem 9.5]:

$$f_a(\psi(z)) = \psi(z^{d+1}). \qquad (*)$$

We say $R_t = \{\psi(re^{2\pi it}) : 1 < r\}$ is the *external ray* with *angle* $t \in \frac{\mathbb{R}}{\mathbb{Z}}$. Then $(*)$ implies $f_a(R_t) = R_{(d+1)t}$. In addition, $\tau_j(K_a) = K_a$ implies $\tau_j \circ \psi = \psi \circ \tau_j$ and thus $\tau_j(R_t) = R_{t+\frac{j}{d}}$.

If $\lim_{r\searrow 1} \psi(re^{2\pi it}) = z \in J_a$, then we say that the external ray R_t *lands* at z. If there exist two distinct rays landing at $z \in J_a$, then we say that z is a *biaccessible point.* By a thorem of F. and M. Riesz [Mi], the point z is a *cut point* of the Julia set J_a, namely $J_a - \{z\}$ is disconnected.

2 Some known results

Very little is known about the topology of the Julia set and the dynamics of polynomials with Cremer fixed points. We have the following results:

- If the origin is a Cremer fixed point, then the Julia set J_a cannot be locally connected [Mi, Corollary 18.6].
- For a generic choice of $|a| = 1$, the origin has the small cycles property, and therefore is a Cremer fixed point [Mi, Theorem 11.13].
- If the origin has the small cycles property, then all critical points c_j cannot be accessible from outside of the Julia set J_a [Ki, Theorem 1.1].

Other results about the semi-local dynamics around Cremer fixed points are refered to [PM]. The following theorem was proved by Pérez-Marco [PM, Theorem 1]:

Theorem 2.1. *Let $f(z) = az + \mathcal{O}(z^2)$ be a local holomorphic diffeomorphism. Assume that the origin is a Cremer fixed point. Let U be a Jordan neighborhood of the origin. Assume that f is defined and univalent on a neighborhood of $\overline{U}$. Then there exists a set H such that:*

- *H is compact, connected and full;*
- $0 \in H \subset \overline{U}$;
- $H \cap \partial U \neq \phi$;
- $f(H) = H$.

In addition, the following holds [SZ, Proposition 2]:

Proposition 2.1. *Assuming the hypothesis in the above theorem, let H be a set given by that theorem. The only point in H which can be a cut point of H is the Cremer fixed point* 0.

3 Main result

Using the preceding results and the following lemma, we can show Theorem 3.1. The method of proof is similar to that of Theorem 3.2.

Lemma 3.1. *Assume that the origin is a Cremer fixed point. Assume that z is a biaccessible point such that $0 \notin \{f_a^{\circ n}(z)\}_{n\geq 0}$ and $c_j \notin \{f_a^{\circ n}(z)\}_{n\geq 0}$ for all j. Then for each j there exist two distinct rays R_{s_j} and R_{t_j} with a common landing point w_j, such that $R_{s_j} \cup \{w_j\} \cup R_{t_j}$ separates c_j from the origin.*

Theorem 3.1. *Assume that the origin is a Cremer fixed point. Assume that z is a biaccessible point. Then $0 \in \{f_a^{\circ n}(z)\}_{n\geq 0}$ or there exists j_0 such that $c_{j_0} \in \{f_a^{\circ n}(z)\}_{n\geq 0}$.*

Remark 3.1. In the above theorem, if the origin has the small cycles property, then $c_j \notin \{f_a^{\circ n}(z)\}_{n\geq 0}$ for all j [**Ki**, Theorem 1.1]. Therefore, the conclusion is just $0 \in \{f_a^{\circ n}(z)\}_{n\geq 0}$.

Finaly, we make mention of the theorem in [**SZ**].

Theorem 3.2. *Let $f_a(z) = z(a+z)$ be a quadratic polynomial. Assume that the origin is a Cremer fixed point. Assume that z is a biaccessible point. Then $0 \in \{f_a^{\circ n}(z)\}_{n\geq 0}$.*

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